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Mathematics News Letter

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To mathematics in general, to the following causes in particular is this journal dedicated: (1) the common problems of grade, high school and college mathematics teaching, (2) the disciplines of mathematics, (3) the promotion of M. A. of A. and N. C. of T. of M. projects.

VOL. 3 BATON ROUGE, LA., APRIL, 1929 NO. 8

THE LAFAYETTE MEETINGS

Joint programs of the Louisiana-Mississippi Section of M. A. of A., the Louisiana-Mississippi Branch of N. C. of T. of M. and the Louisiana Academy of Sciences were carried out in a most admirable manner at Lafayette, Louisiana, on the 12th and 13th of this month.

The names of the new officers of both the Section and the Council are found on the title page of this issue of the News Letter. The following measures were voted upon favorably by the two organizations acting jointly: (1) In the future the annual meetings of the two bodies shall be held on the first or on the second week-end of the month of March. (2) An editor to represent grade mathematics and its teaching shall be added to the editorial staff of the News Letter, Miss Dora M. Forno, of the New Orleans Normal School mathematics department to fill this position. (3) The office of Business Manager of the News Letter shall be created, such office to be filled by the Chairman of the Section. (4) The chief function of the Business Manager shall be to put and to continue in operation substantially the following scheme for developing adequate financial support of the Mathe-

matics News Letter in our Louisiana - Mississippi territory. Every large town of Louisiana and Mississippi—except as noted below—will be the made the center of a campaign district consisting of the central parish or county and parishes or counties which are adjacent to it. Some teacher of mathematics, whether high school or college, it matters not, will be appointed by the Chairman to be a superintendent of the district, place of residence of the superintendent to be, preferably, in the large-town center. The duty of this superintendent will be the single if not simple one of putting into operation ways and means by which every mathematics teacher in his district may be induced to subscribe to the News Letter. He will have full power to organize a body of helpers as he may see fit in order to achieve these ends. In the case of cities, as New Orleans, Shreveport or Jackson, the Chairman will use his discretion in making a district of the one parish, or the one county, containing the city. No town will of course be chosen as a center of operations in a district to be formed unless its geographical situation is the most favored one from the view-point of affording the most convenient opportunities of the district superintendent to have, if necessary, personal contacts with the mathematics teachers. (5) The next meeting place of the two mathematics bodies will be Cleveland, Mississippi, with the Mississippi Delta State Teachers College doing the part of host.

—*S. T. S.*

PROFESSOR PALACIOS' RESEARCH IN METHODS

It appears from reports of the work of Miss Adelia Palacios, professor of mathematics in the National School of Teachers and the University of Mexico City, that she has made a thorough job of the search for and application of the underlying principles governing the psychology of learning mathematics.

The method outlined will economize much time which can be profitably spent in individual pursuits, mathematical or otherwise. It is claimed that a child can learn the multiplication tables in five weeks and at the age of six. By this method the fundamental operations of arithmetic, algebra, geometry and even calculus will be made proportionately easy.

We cannot make mathematics easy but we can make it less difficult by the application of the principles of all the ologies that underly either the learning or the teaching of the subject.

We await with interest the coming of a new method of teaching mathematics that will arouse in the pupils a keen desire to do his best work, and that will reduce the learning time of the fundamentals of arithmetic. We will welcome with open minds Miss Palacios' theories and stand ready to accept all that will be helpful in the many problems of both the teacher and the pupil in the study of mathematics.

—D. M. F.

ON THE NATURE OF MATHEMATICAL REASONING

(Third Paper)

Bertrand Russell, one of the two or three greatest living mathematical philosophers, identifies logic and mathematics. There are technical mathematicians of the highest rank who deny that they are identical. Analysis would probably show that such contrariety of conclusion arises from differences in accepted definitions of mathematics. In the lay mind mathematics is inseparably bound up with notions of quantity, number and measurement. If this is admitted as an essential mark of mathematics then logic and mathematics are not identical. On the other hand there are important branches of mathematics which are not essentially related to quantity or number, such as those branches which deal primarily with the notions of projection, order, form, etc. It may safely be granted that if there are important differences in the two fields such differences must be due to existing differences in the materials to which the two processes are customarily applied. It is conceivable that a logical process may not be mathematical. It is inconceivable that a mathematical process should not be logical. The distinction suggests that mathematics might be termed a form of applied logic—a characterization in harmony with the frequently used language "Mathematics is a logical consistency." It is natural at this point that we take more than a passing glance at that characteristic of a "perfected inferential process" which we have described as

"waiving the matter of the truth or the falsity of the basic hypothesis or hypotheses."

Whenever the conditions which affect the series of logical deductions are of a kind which demand that the basic assumptions shall be true, not false, such a series of deductions automatically fails of classification as pure mathematics, though, possibly, it may be classifiable as a type of applied mathematics. For example, it is not a matter of indifference to the astronomer whether the deductions which make up a science of celestial mechanics flow from Newton's gravitation hypothesis or from Einstein's assumption of the relativity of motion. The astronomer's primary demand is that they shall be consistent with reality. He demands more than that they shall be consistent with a received hypothesis. To him the vital thing is that they shall be consistent with the truth of the motions of the heavenly bodies. Thus the logical deductions which make up a given system of physics constitute a form of applied mathematics. On the other hand to the pure mathematician it is a matter of indifference whether the Euclidean, the elliptic or the hyperbolic hypothesis be the received basis of a system of geometry. His vital interest is in the search for the number and nature of the deductions which characterize any chosen hypothesis. He does not care whether two straight lines drawn perpendicular to a third one will, upon sufficient extension, remain parallel, intersect, or diverge. He is only vitally concerned to know the nature of the deductions peculiar to each hypothesis, the differences which exist among them, their contrasted or their common properties.

Because of the arbitrary character of a mathematical hypothesis it has been customary, from the most remote times, to refer to mathematics as the science which deals with ideal conceptions, as, for example, points, lines, surfaces, irrational numbers, imaginary numbers, infinitely small, infinitely large quantitis. It accounts for the diversity of algebraic systems which have been created in the last century—such diversity having, largely, been caused by the deliberate omission of one or more of the basic assumptions which characterize ordinary algebra.

In closing this, for the present, the last of our series of papers on the nature of mathematical reasoning, it is fitting to

remark that we do not pretend to have more than begun a study which is susceptible of considerable extension.

—*S. T. S.*

NOTES ON THE ORIGIN AND USE OF DECIMALS

By DORA M. FORNO
New Orleans Normal School

Decimal fractions were introduced so gradually that historians differ slightly in assigning their origin to a specific person. To a student of the history of the development of the subject it appears that decimal fractions were staring mathematicians in the face for a long time before their significance was realized. A sudden stroke of good fortune revealed the full significance of decimal fractions.

The invention of decimal fractions is generally credited to Simon Stevin of Bruges in Belgium (1548-1620). In 1584 he published in Flemish an interest table and the working out of this table undoubtedly led to the discovery of the new decimal system. The only thing that was lacking was a suitable notation. In 1585 Stevin published his La Disme, a French work on mathematics, in which decimal fractions were explained. In place of our decimal point he used a cipher and to each place in the decimal fraction was attached the corresponding index. Thus 3.576 was written: $\overset{0}{3}\;\overset{1}{5}\;\overset{2}{7}\;\overset{3}{6}$ or in another form, which owing to difficulties in setting it up in type is here omitted. Stevin explained in very definite terms the advantages to be derived from the use of decimal fractions in all operations and those to be derived from the decimal subdivision of the units of length, area capacity, value, etc. To us who have been using the decimal fractions so naturally, it appears that the extension of the Hindu system of notation to the right of units was a very simple thing and that mathematicians were very short-sighted not to have discovered decimal fractions long before they did.

The earliest indications of the decimal idea were shown by Fineus, a French mathematician. In extracting the square root of 10, be extracted the square root of 10,000,000 and obtained

3162 and, instead of using the same symbolism to express the square root of 10 as 3.162, he translated the result into sexagesimal fractions with 60 as a base. The square root of 10 was expressed as 3 9′ 43″ 12‴, or $3+\frac{9}{60}+\frac{43}{3600}+\frac{12}{216000}$

Bürgi, a Swiss by birth, used the decimal fractions after 1592, and used a zero in unit's place as a sign of separation. In 1603 Johann Beyer published his Logistica Decimalis and assumed the invention of the decimals as his own. He wrote 5.5367 thus: $\overset{0}{5}.\overset{\mathrm{I}}{5}.\overset{\mathrm{II}}{3}.\overset{\mathrm{II}}{6}.\overset{\mathrm{IV}}{7}.$ or .0005 thus: $\overset{\mathrm{IV}}{5}$

Peacock credits Napier with the introduction of the decimal point. In 1617 Napier published his Rabdologia containing a treatise on decimals wherein the decimal point is used in one or two instances, and in 1616, in the English translation of Napier's Descriptio by Edward Wright, the decimal point appears in the logarithmic tables. Other symbolisms were in use. Oughtred in 1631 in an English arithmetic designated .56 thus, $\underline{|56}$

George Andres Böckler of Nurnberg, 1661, used the comma in place of the point, the method used by the Germans.

Henry Briggs, a professor of geometry in Gresham College, London, called on John Napier, the inventor of logarithms, in his Scottish home to pay homage to this worthy inventor and as a result of the visit and exchange of ideas developed the "Briggsan Logarithms," one of the most practical applications of decimal fractions. Briggs underlined the decimal figures thus: 25 . 379 was written in the form of $25\ \frac{379}{1}$. Ball credits Briggs with the introduction into arithmetical processes of the decimal notation for fractions.

Decimal fractions are defined as all fractions which have a decimal number for the denominator whether the denominator is expressed or understood. It is not adequate to define a decimal fraction as a fraction whose denominator is not written but is some power of 10, for 5/100 is just as much a decimal fraction as .05. The method employed in changing a decimal fraction from the form of the common fraction to the new

symbolism is to point off in the numerator as many decimal places as there are ciphers in the denominator.

It was not until the eighteenth century that decimal fractions found much footing and not until the nineteenth century that their use became general.

The question of the sequence of common and decimal fractions in the teaching of the subjects is one that has been greatly discussed. Someone has remarked that "logically the decimal fraction comes first, because it grows naturally out of our number system." Historically and practically the need for our so-called common fractions comes first and hence should be taught first. A Prussian educational decree of 1872 put the decimal fractions first, but the Prussian experiment produced doubtful results and other cities continued following the older plan.

The general conclusion is that an elementary treatment of simple fractions has first place and that they are introduced as the need for them arises and that long before the pupils are introduced to the difficulties of the operations of common fractions problems in United States money have fully acquainted them with the decimal notation and the simple operations therewith.

Can all fractions be reduced to decimal fractions? They cannot. If the denominator have any other prime factor except 2 and 5, the fraction cannot be converted into a decimal fraction. The denominators must be made up of 5×2, 2×2, 5×5, $5\times5\times5$, $5\times5\times2$, etc. All fractions can, however, be expressed approximately in terms of decimal fractions to within a slight degree of error which will not materially affect the result. A difference of one thousandth of an inch in measurement is an immaterial difference.

Because of the extensive use of decimal fractions at the present day in reading speedometers, cyclometers, statistical reports, etc., it is most important that pupils have clear ideas of the meaning of decimal fractions and be able to interpret them rightly. Ask the average pupil about how far you would walk if you walked .495 of a mile and note the variety of answers. Another interesting question would be to ask him to arrange in order of size the following numbers: .34 , .6 , .143 , .0789 and note results.

The operations of addition and subtraction of decimals and multiplication and division of decimals by a whole number present no difficulties to the average pupil, because they are so closely related to the problems of United States Money. Multiplication of a decimal by a decimal does not present any serious difficulty, as the fundamental principles of a tenth times a tenth equals a hundredth, etc., lays the foundation for the rule for pointing off in the product. A study of the relative merits of the two methods of placing the decimal points in the division of decimals has been made. The common rule in the older arithmetics is: "There are as many decimal places in the quotient as those in the dividend exceed those in the divisor." The Austrian method is: "First render the divisor an integer by multiplying both the dividend and the divisor by 10 or some power of 10." In the light of the data secured it seems that the Austrian method secures better results and should replace the old method in all teaching of decimals. A series of carefully arranged examples for locating the decimal point in the quotient is a desirable introduction for deriving the statement of the rule for the location of the decimal point, and a checking of results to notice always that the number of decimal places in the divisor and quotient together equal the number of decimal places in the dividend prevent the possibility of errors.

SCIENCE IN MULTIPLYING FRACTIONS.*

C. D. SMITH
Louisiana College

The process of multiplying fractions is a science which can be approached to the best advantage thru some scientific plan. One approach to the problem may be made first by stating briefly the essential facts and following with a planned practice schedule. The following will illustrate the notion of the writer which has the support of results obtained by experiment.

Numbers constitute the best known method of designating size or quantity. The classes of negative and positive numbers indicate distances in opposite directions or quantities in the opposite sense. We may represent the system of integers with

one as unit by the diagram

----- -2, -1, 0, 1, 2, ----.

The fraction is a device for describing quantities in terms of units less than one. For example, 0, ½, 1, represents by ½ the fact that one is to be separated into two equal parts. Now with a fractional system we understand that

$5\times1/2=5/2$ (five halves) just as $5\times1=5$ (ones)

Hence for the first class of multiplications we have Type I which we represent by $(a\times b/c)$.

We may find common measures of different quantities by suitably changing the units. For example when

$1/2\times1/3=1/6$

the unit suitable for both quantities is given by 1×1 over 2×3. Hence we have as a second class of multiplications Type II which we represent by $(d/e\times f/g)$.

The next idea is that of the mixed number or improper fraction. Suppose we represent certain distances from zero to two by the diagram.

O, 1/2 1 1½ 2.

Now suppose we wish to represent the distance from zero to a point half way between 1 and 2. If we mix the units for measuring different parts of the distance we write 1½, or if we use ½ as a unit the distance is 3/2. This gives Type III multiplications of mixed numbers by integers which we represent by $(h\times i\ j/k)$. We may have multiplications of fractions and mixed numbers which we represent by Type IV, $(l/m\times n\ p/q)$. Finally for Type V $(r\ s/t\times u\ v/w)$ we have multiplications of mixed numbers.

In multiplication practice we learn the advantages of change to the largest unit possible, reduction, and cancellation. As a scientific scheme of planned practice we submit the following using only positive numbers.

Type Variations

Type	Unit A	Units Combined B	Cancelled Factor C	Factor Common D
I $(a\times b/c)$	$2\times1/3$	$3\times2/5$	$3\times1/6$	$4\times5/6$
II $(d/e\times f/g)$	$½\times1/3$	$½\times3/5$	$½\times4/7$	$¾\times10/13$

III $(h\times i\ j/k)$	2×1 1/3	2×3 2/3	2×3 3/4	4× 3/10
IV $(l/m\times n\ p/q)$	¼×1 1/2	¾×2 1/2	4/7×2 1/2	6/13×2 1/4
V $(r\ s/t\times u\ v/w)$	1½×2 1/4	2½×3 2/3	2½×2 3/10	2 2/3×5 1/6

Considering the type variations as elements of a matrix we may plan practice proceding from the simpler to the more complicated in three ways namely;

1. By columns beginning on left.
2. By rows beginning at the top.
3. By diagonals beginning with I A; II A, I B; III A; II B, I C; etc.

The third plan is to be preferred because the types and variations recur at intervals as you proceed. Give five of each type variation thus covering 100 practices. Test for weak spots and practice further as indicated.

It may be said in conclusion that trial followed by measurement will prove that this method excels any chance method in multiplication. This discussion is intended to illustrate what may be done by scientists in planning scientific processes.

*Read before the Louisiana Academy of Sciences, April, 1929.

THE EGYPTIANS AS PURE MATHEMATICIANS

By IRBY C. NICHOLS
Louisiana State University

The civilization upon the banks of the Nile is rated as one of the great civilizations of ancient times. If so, its mathematics should have been correspondingly great, for mathematicians claim that **"the mathematics of a people furnish an accurate index as to the degree of their civilization."** Yet our historians seem to hold to the view that the only mathematics the Egyptians had was a set of empirical rules intended for purely practical purposes.

Unquestionably such a view is entirely too narrow: First, it is most improbable that ancient Egypt could not boast of her scholars who sought learning, who studied mathematics for the sake of mathematics, and who held the scholar's in-born aver-

sion to studying for the sake of pecuniary returns, because Egypt certainly could boast of the splendor and brilliance of her courts, of her rich and aristocratic classes; and, among these, certainly there must have been those who had plenty of leisure for studying for their own personal pleasure and interest.

Second, it must be remembered that, in the days of Egypt and, indeed, until Euclid's time, a mathematician was a philosopher and a philosopher was a mathematician. So the philosopher of Egypt most naturally included mathematics in his studies.

But beyond these first and second reasons there is the more convincing evidence of that oldest of arithmetics in existence, the Ahmes, or Rhind papyrus, estimated to have been written more than thirty-five hundred years ago, and most probably based on other texts and sources still older by many centuries. Repeated mention is made of Egypt's geometry as treated in this work, but attention is seldom called to her arithmetic calculations, particularly to those forms called fractions, which, while they are certainly practical in their need, still furnish illustrations which in themselves appear to be merely theoretical. The primary purpose of this brief article* is to submit a study of these forms, and leave it to the reader to judge whether the Egyptians are entitled to more credit in mathematics than is commonly accorded them.

In form, the Egyptians used unit fractions—fractions with a constant numerator 1: It is by these fractions that Egyptian influence is traceable through Greek, Roman, Arabic, and Middle-age European mathematics. The one exception to this form was 2/3. Their notation likewise was a single notation: ½, 1/3, 1/4, and 2/3 were expressed by special symbols, while other fractions were expressed by writing their respective denominators with a dot above, as $\dot{7}$ for 1/7.

Abstractly, **the Egyptians had other fractions**, expressed always in terms of unit fractions. Instead of 2/5 they wrote 1/3 1/15; instead of 2/13 they wrote 1/8 1/52 1/104. For the sake

*This article is based upon notes made from Eisenlohr's translation (second edition, 1891). The particular copy used was from the library of the late Professor A. Ziwet, of the University of Michigan. Eisenlohr's work is in German, of course, but an excellent English translation of the work of Ahmes is being published.

of convenience, tables are given by Ahmes for converting fractions of the form $\frac{2}{2n+1}$, $n=1, 2, \text{- - -} 49$, into fractions. These tables are probably the results of the accumulated experience of earlier mathematicians and not of any general formula, except for the special case when $2n+1$ is a multiple of 3. Then the rule is given to multiply the denominator by 2 and by 6 respec tively; that is, $2/3n=(½+1/6)(1/n)$. Thus $2/9=½\cdot1/3+1/6\cdot1/3=1/6+1/18$.

As a technical term, a **common denominator** of two or more fractions is not specifically mentioned, but **solutions of examples reveal the conception of such in its practical application.** With unit fractions, numerators of fractions reduced to a common denominator could be found very readily.

Ahmes solved examples involving all four of the fundamental operations with fractions. **Subtraction is treated as additive completion of the subtrahend to the minuend, and division as multiplicative completion of the division to the dividend.** Numerous examples are solved, but no formal rules are given. As in the multiplication of integers, a number is doubled, redoubled, and then again redoubled until the addition of certain of these multiples gives the desired product; for example, to multiply by 13, a person would add 1, 4, and 8 times the multiplicand, and thus obtain 13 times the multiplicand. In fractions, the same method is pursued. The following examples furnish an instance in which both multiplication and division are required in the same exercise: To add to ¼ and 1/28 their ½ and ¼; that is, solve $(¼+1/28)+(½+¼)(¼+1/28)$. The text (examples 7, p. 36) presents the solution in the following form:

	¼	1/28	¼	1/16	1/112
	7	**1**		**1**	**½ ¼ ¼**
½	1/8	[1]/56			
				together ½.	
	3 ½	**½**			

Clearly ¼ and 1/28, reduced to a common denominator of 28, give as numerators 7 and 1 respectively. Dealing with these numerators, ½ of ¼ and ½ of 1/28 become 3 ½ and 1/2 respectively, and ¼ of ¼ and ¼ of 1/28 become 1 ½ ¼ and ¼ respectively. Adding 7, 1, 3 ½, ½, 1½, ¼ and ¼, the

sum is 14; that is 14/28 or ½ as shown in the solution above. In the text the numbers to be added are written in red color, while all other terms are in black.

An example in subtraction is the following one in which the student finds the difference between 1 and the sum 2/3 and 1/15 (example 21, p. 39):

You are told to complete

2/3 1/15 to 1

10 1 together 11 remainder 4 multiply 15 to find 4

15 1/15 1

1/10 1 ½ together 4

1/5 3 Hence 1/5 and 1/15 are to be added to obtain 4/15."

Notice the use of 1/10 of 15. From the Egyptian manner of writing it was easy to see that 1/10 of 15 is 1 ½. The author checks his result by addition:

2/3, 1/5, 1/15, 1/15, when added, give 1.

ON THE ABSOLUTE MAXIMUM OF A FUNCTION ON A REGION

By H. L. SMITH
Louisiana State University

In the calculus considerable attention in given to relative maxima and minima of functions, but little is said about absolute maxima and minima. Yet it is the latter which occur more often in applications. In this note a method is given which is sufficient for simple cases.

Consider first a function $f(x)$ which is continuous and has a continuous first derivative $f^1(x)$ on the interval I: $a \leq x \leq b$. It is known that $f(x)$ has an absolute maximum on I which is attained for some value of x on I; we seek a method for determining such a value of x. Suppose $f^1(x)$ vanishes on I at a finite number of points at most, and let $c_1, : : , c_n$ be the set of all such points at which $f^1(x)$ changes sign from positive to negative. Then the absolute maximum of $f(x)$ on I is clearly the largest of the numbers $f(a), f(c_1), \ldots, f(c_n), f(b)$.

Consider now a function $f(xy)$ of two variables, continuous

and having continuous first partial derivatives $f_1(xy)$, $f_2(xy)$ on the region R defined by the inequalities,

$$a \leqq x \leqq b, \quad u(x) \leqq y \leqq v(x),$$

where $u(x)$, $v(x)$ are continuous for $a \leqq x \leqq b$. This problem reduces to the preceding one as follows. Let $g(x)$ be the absolute maximum of $f(xy)$ as y varies over the interval $u(x) \leqq y \leqq v(x)$, x being fixed. Then let M be the absolute maximum of $g(x)$ on $a \leqq x \leqq b$. It is easily seem that M is the absolute maximum of $f(xy)$ on R.

This method is illustrated by the solution of the following

Problem.* Find the absolute maximum of the ratio, S^2/K, of the square of the semi-perimeter to the area of a triangle A B C, the triangle being subject to the restriction that each of the numbers tan ½A, tan ½B, tan ½C is at least d, a fixed postive number.

We have (Cf. this **News Letter**, vol. 3, no. 7, p. 22)

$$S^2/K = \text{ctn } \tfrac{1}{2}A \ \text{ctn } \tfrac{1}{2}B \ \text{ctn } \tfrac{1}{2}C$$
$$= [x+y]/[xy(1-xy)],$$

where $x = \tan \frac{1}{2}A$, $y = \tan \frac{1}{2}B$. Suppose the notation such that $A \leqq B \leqq C$. Then $C = 180° - (A+B)$,

$$\tfrac{1}{2}A \leqq \tfrac{1}{2}B \leqq 90° - \tfrac{1}{2}(A+B)$$

so that

$$d \leqq x \leqq y \leqq (x+y)/(1-xy),$$

which may be written in the form

$$(1) \quad d \leqq x \leqq y \leqq \sqrt{x^2+1} - x$$

Moreover since $A \leqq 60°$,

$$(2) \quad d \leqq x \leqq 1/\sqrt{3}$$

Hence our problem is to find the absolute maximum of the function

$$f(xy) = [x+y]/[xy(1-xy)]$$

over the region defined by the inequalities (1), (2).

On differentiating $f(xy)$ partially with respect to y, we obtain

$$f_2(xy) = [(y+x)^2 - (x^2+1)]/[y^2(1-xy)^2].$$

On account of (1), $f_2(xy) < 0$ on the interior of R. Hence $g(x)$, the absolute maximum of $f(xy)$ on the interval (1), is

$$g(x) = f(x, x) = 2/[x(1-x^2)].$$

*A more elementary solution of an equivalent problem is given by S. T. Sanders on p. 18 of this issue of the **News Letter**.

Also

$$g^1(x)=2(3x^2-1)/[x^2(1-x^2)^2].$$

For the interior of the interval (2), $g^1(x)<0$ and hence M, the absolute maximum of $g(x)$ on (2), is

$$M=g(d)=2/[d(1-d^2)].$$

This is the absolute maximum of $f(xy)$ on R and is attained when $x=y=d$.

NOTE ON FACTORIZATION IN CONNECTION WITH A CERTAIN DETERMINANT

By S. T. SANDERS
Louisiana State University

The realization on the part of research workers in number theory that certain algebraic numbers may be resolved into prime factors in more than one way was the signal for the beginning of the construction of one of the most elegant branches of modern mathematics, namely, the theory of ideal numbers. What are the prime factors of 26? If we answer in terms of ordinary number theory we say: 2, 13. If we answer in terms of the domain $k\sqrt{-1}$, k rational, we say: $5+\sqrt{-1}$, $5-\sqrt{-1}$.

On page 126 of Dickson's First course in the Theory of Equations a problem is proposed, namely, to show that

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = (a+b+c)(a+bw+cw^2)(a+cw+bw^2)$$

By repeated application of a familiar law of the determinant, namely, that the value of the determinant is unaltered if to all the elements of any row (or column) is added the product of an arbitrary constant k by the corresponding elements of any other row (or column), we have

$$(1) \quad \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \begin{vmatrix} a+b+c, & b, & c \\ a+b+c & a, & b \\ a+b+c & c, & a \end{vmatrix}$$

$$(2) \quad = (a+b+c) \begin{vmatrix} 1, & b, & c \\ 1 & a, & b \\ 1 & c, & a \end{vmatrix}$$

The passage from (1) to (2) is made by a choice of k from the domain $R(a, b, c)$, R standing for a rational operation. But the above cited law of the determinant does not depend upon a particular choice of k, that is, k may be chosen arbitrarily.

For brevity denote this determinant law by $\llcorner$, its operation with k upon the elements of row r_k to modify the elements of row r by

$\llcorner(k, r_k)r_s$, or $\llcorner(k, c_k)c_s$, where c represents a column

Then if $\llcorner(w, r_3)r_2$ is applied to the determinant of (2), there results

$$(3)\qquad = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1+w, & a+cw, & b+aw \\ 1 & c & a \end{vmatrix}$$

Again, $\llcorner(w, r_2)r_1$ applied to (3) results in

$$(4)\qquad = (a+b+c)\begin{vmatrix} 1+w+w^2, & aw+cw^2+b, & bw+aw^2+c \\ 1+w, & a+cw, & b+aw \\ 1 & c & a \end{vmatrix}$$

If now, as was intended in the proposed problem, we assign w to an imaginary domain by letting it be an imaginary cube root of unity, (4) becomes

$$(5)\qquad = (a+b+c)\begin{vmatrix} 0, & aw+cw^2+b, & bw+aw^2+c \\ 1+w, & a+cw & b+aw \\ 1 & c & a \end{vmatrix}$$

On account of type-setting difficulties, w is here used instead of the usual Greek symbol.

Multiplying the third column of (5) by w and multiplying the second column by $1/w$, (5) becomes

$$(6)\qquad =(a+b+c)\begin{vmatrix} 0 & a+bw^2+cw, & a+bw^2+cw \\ 1+w, & aw^2+c & bw+aw^2 \\ 1 & cw^2 & aw \end{vmatrix}$$

from which we easily have

$$(7)\qquad \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} =(a+b+c)(a+bw+cw^2)(a+bw^2+cw),$$

by using the familiar properties of w.

If a, b, c be arbitrary arithmetic integers, the value of the determinant, or of $a^3+b^3+c^3-3abc$, is some arithmetic integer. Denoting it by R, we may write $R=p_1 : . . p_k$, where the p's

are its arithmetic prime factors. If the prime factors of $(a+b+c)$ are $c_1 \ldots c_s$, we should have

(8) $p_1 \ldots p_k = (c_1 \ldots c_s)(a+bw+cw^2)(a+bw^2+cw)$

the left member showing the prime factors of the determinant in the domain rational, the right member showing its factors in the domain of an imaginary cube root of unity.

We make the following comments:

(a) Property $\llcorner$ of the determinant, as defined above, was effective in developing this double factorization.

(b) Every integer R (arithmetic) which is susceptible of expression in the form $a^3+b^3+c^3-3abc$ has prime factors in the domain of an imaginary cube root of unity and, moreover, they are immediately obtainable by (7).

(c) It should be an interesting investigation—this writer has not undertaken it, possibly it may have already been done—to determine if analogous formulae exist in respect to n^{th} roots of unity, n any positive integer.

(d) **Example.** Let $a=2$, $b=4$, $c=9$

Then $a+b+c=5\cdot 3$

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \begin{vmatrix} 2 & 4 & 9 \\ 9 & 2 & 4 \\ 4 & 9 & 2 \end{vmatrix} = 585$$

By (8), $585=5\cdot 3\cdot 39=5\cdot 3(2+4w+9w^2)(2+4w^2+9w)$

Whence $39=3\cdot 13=(2+4w+9w^2)(2+4w^2+9w)$

In his interesting article of the present issue "The Egyptians as Pure Mathematicians," Professor Nichols has reminded us that in the old Egyptian times, "a mathematician was a philosopher and a philosopher was a mathematician." The implication is plain—and a correct one—that in present times this is not always true. The mathematician who doubts it has only to read such statements as the following from Henri Bergson's "Time and Free Will" and then reflect: Bergson is no mathematician, and, since he here philosophizes about mathematics, if his mathematics is not good, how can his philosophy be? The passages from Bergson are as follows:

"It is not enough to say that number is a collection of units; we must add that these units are identical with one another or

at least that they are assumed to be when they are counted - - - On the other hand, as we fix our attention on the particular features of objects or individuals we can of course make an ennumeration of them but not a total - - - - For we can confidently assert that 12 is half of 24 without thinking either the number 12 or the number 24 - - - - It cannot be denied that the formation or construction of a number implies discontinunity."

PROBLEMS

Proposed by H. L. Smith: Show that in any triangle

$$\frac{s^2}{k} \leqq \frac{(1+\sqrt{1-e^2})^2}{e\sqrt{1-e^2}} < \frac{8}{e}$$

where s is its semi-perimeter, k its area and e is a positive number at most equal to the smallest of the sines of the angles of the triangle.

***Solved by S. T. Sanders:** In the March issue of the News Letter in connection with the solution of a previously proposed problem we showed that

$$\frac{s^2}{k} = \sqrt{\frac{(1+\cos A)(1+\cos B)(1+\cos C)}{(1-\cos A)(1-\cos B)(1-\cos C)}}$$

where $A \leqq B \leqq C$.

Another form of (1) is

$$\frac{s^2}{k} = \sqrt{\frac{(1+\cos A)(1+\cos B)(1-\cos(A+B)}{(1-\cos A)(1-\cos B)(1+\cos(A+B)}}$$

If in (2) we let $A=B$, we get

$$\frac{S^2}{K} = \frac{1+\cos A}{1-\cos A}\sqrt{\frac{1-\cos 2A}{1+\cos 2A}}$$

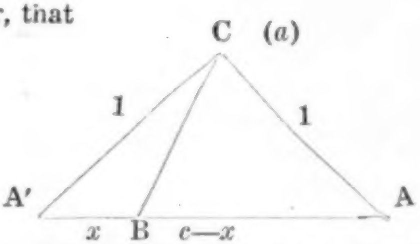
$$= \frac{1+\cos A}{1-\cos A} \quad \frac{\sin A}{\cos A}$$

or

$$\frac{S^2}{K} = \frac{(1+\sqrt{1-\sin^2 A})^2}{\sin A\sqrt{1-\sin^2 A}}$$

We now show that $\dfrac{s^2}{k} \leqq \dfrac{S^2}{K}$, or, that

(1) $$\frac{s^2}{S^2} \leqq \frac{k}{K}$$



Let S and K belong to triangle A C A', s and k to triangle B C A, as seen in Fig. (a). Let sides, A C=A' C, A'A, A'B, B A, be, respectively 1, c, x, $c-x$.

Then $\dfrac{k}{K}=1-\dfrac{x}{c}$ and

$$\frac{s^2}{S^2}=\frac{(1+c-x+\sqrt{1-cx+x^2})^2}{(2+c)^2}$$

Thus to establish (1) it is sufficient to establish

(2) $$\frac{(1+c-x+\sqrt{1-cx+x^2})^2}{(2+c)^2} \leqq 1-\frac{x}{c}$$

Since B $\leqq$ C, if we think of B as increasing from the value A', corresponding to which x increases from 0, and k decreases from K, A'=A being fixed, the lower bound of $c-x$ is seen to be $c-x=1$, from which it follows that

(3) the upper bound of x is $c-1$.

$(0 \leqq x \leqq c-1)$

The truth of (2) will be established by showing that if (2) is assumed to be false (3) will be contradicted.

If (2) is false,
then,

$$(2+c)^2(1-x/c) \leqq (1+c-x)^2+2(1+c-x)\sqrt{1-cx+x^2}+1-cx+x^2$$

(4) or, $c^2+c+c^2x-(cx^2+2x+cx) \leqq (c+c^2-cx)\sqrt{1-cx+x^2}$

The right member of (4) is evidently positive. Since c^2x is greater than cx^2, its left member is positive if $c^2+c-(2x+cx)$ is positive.

Since for $x=0$ the value of this expression is positive, it is clear that as x increases its value must steadily decrease from

the positive number (c^2+c). But x is never greater than $c-1$.

Placing $c-1$ for x in $c^2+c-(2x+cx)$, we have

$$+c^2+c-2c+2-c^2+c=2$$

Thus the minimum value of $c^2+c-(2x+cx)$ is positive, so that the left member of (4) is positive and the sign of inequality will be unchanged if (4) should be squared.

Squaring (4), if (4) is true, we should have

$$c^2x^4+x^3(4c+2c^2-2c^3)+x^2(c^4-4c^3-5c^2+4c+4)+x(2c^4-6c^2-4c)$$
$$+c^4+2c^3+c^2 \leqq c^2x^4+x^3(-2c^2-3c^3)+x^2(2c^2+4c^3+3c^4)$$
$$+x(-2c^2-3c^3-2c^4-c^5)+c^4+2c^3+c^2,$$ that is,

(5) $x^3(c^3+4c^2+4c)+x^2(-2c^4-8c^3-7c^2+4c+4)+x(c^5+4c^4+3c^3-4c^2-4c)\leqq 0$

As x and its coefficient are positive, we may divide (5) by $x(c^3+4c^2+4c)$ without affecting the inequality sign. Thus if (5) is true,

(6) $x^2+x(-2c+1/c)+c^2-1\leqq 0$

Case 1. If $x^2+x(-2c+1/c)+c^2-1=0$,

$$x=c-1/c, \text{ or } c$$

The first of these values is consistent with (3) for the minimum value of c, namely $c=1$ and hence if this value of c be assumed (3) is not contradicted. But for $c<1$, $x=c-1/c$ is greater than $c-1$ and (3) is contradicted.

Cases 2. If $x^2+x(-2c+1/c)+c^2-1<0$,

let $x^2+x(-2c+1/c)+c^2-1=-n$, n positive

Then $x=2c-1/c\pm 1/2\sqrt{1/c^2-4n}$

Case 2a. If $n>1/4c^2$, x is imaginary

Case 2b. If $n=1/4c^2$,

$$x=(c-1/2c)>c-1$$

Case 2c. If $0<n<1/4c^2$

$x=c-1/2c\pm d$, where $0<d<1/2c$,

and where $(c-1/2c\pm d)>c-1$

Thus we have shown that in all cases the assumption that (2) is correct leads to a contradiction of (3) except in the case where $c=1$, or $x=0$, that is, it leads to the condition $x\geqq c-1$, instead of $x\leqq c-1$.

Hence (2) is false and we have

$$(7)\quad \frac{s^2}{k} \leqq \frac{(1+\sqrt{1-\sin^2 A})^2}{\sin A\sqrt{1-\sin^2 A}}$$

It remains to show if $e<\sin A$, A the smallest angle of the triangle, that

$$(8) \quad \frac{s^2}{k} \leqq \frac{(1+\sqrt{1-e^2})^2}{e\sqrt{1-e^2}}$$

The right member of (7) may be expessed in the form

$$(9) \quad \text{Sec A} \frac{1+\cos A}{\sin A} + \frac{1+\cos A}{\sin A}$$

The proof of (8) is accomplished, if we show that as A is decreased (9) increases in value.

It is evident that as A is decreased $\frac{1+\cos A}{\sin A}$ increases since the numerator increases while the denominator decreases.

As A is at most 60°, 2 sec A>1

Thus if d should represent an amount of increase in the value of $\frac{1+\cos A}{\sin A}$ corresponding to a decrease of the angle from A to B, the corresponding amount of decrease in sec A is < 1, whence the corresponding maximum amount of decrease in the first term of (9) is $<d$. Hence the increase of the second term of (9) is greater than the decrease of the first term, so that (9) increases as A decreases.

This proves (8).

That $\frac{(1+\sqrt{1-e^2})^2}{e\sqrt{1-e^2}}$ is less than $\frac{8}{e}$ becomes evident when, it is remembered that $(1+\sqrt{1-e^2})^2<4$. Since e is not zero $\sqrt{1-e^2} \leqq \frac{1}{2}$. From this it follows that

$$\frac{(1+\sqrt{1-e^2})^2}{e\sqrt{1-e^2}} < \frac{4}{e.\frac{1}{2}}$$

or

$$\frac{(1+\sqrt{1-e^2})^2}{e\sqrt{1-e^2}} < \frac{8}{e}$$

*Purposely solved by inequality theory and elementary method. Briefer ways of solving it by the calculus exist. See H. L. Smith's solution in this issue of the Letter.

A MATHEMATICAL NIGHTMARE

The secants flutter all about,
  The scarlet tangents sing:
The blooming polygons are pink,
  And spheres are on the wing.

Fierce propositions roam the woods,
  And cosines fill the air
With music sweet; bright hexagons
  Are growing everywhere.

The octagon sits on her nest
  To keep the quadrant safe
And warm, until it hatches out
  A quadrilateral waif.

When Fall is here, and love is warm;
  Matriculations mate;
The quadrant to the sextant sings,
  And rhombuses rotate.

—Northwestern Purple Parrot.

AN INVITATION

Competent and experienced mathematical workers are invited to furnish contributions to the Letter on such topics as the following:

Mathematics and the principle of election in a liberal arts college program.

The mathematical mind and the non-mathematical mind—is the difference real?

Methods of motivating interest in mathematics.

Sketches of successful mathematicians.

Mathematics and school administrations.

Personality in mathematics teaching.

Review of important articles in the Mathematics Teacher.

Review of important articles in the American Mathematical Monthly.

News notes from college and high school mathematics departments of Louisiana and Mississippi.

S. T. SANDERS IN ACCOUNT WITH MATHEMATICS NEWS LETTER OVER PERIOD FROM MARCH, 1928 TO MARCH, 1929

Expenses

Printing eight issues of News Letter	$414.60
Cost of mailing out	46.97
Cost of subscription slips and statements	4.50
TOTAL	$466.07

Receipts

Donations from L. P. I., $18.00; Centenary $18.00; University of Mississippi, $18.00 and forwarded for News Letter expenses	$ 54.00
Subscriptions (Since Jackson meeting)	141.00
Subscriptions renewed	23.00
*Other amounts from Secretary Cole	41.35
Individual donations	33.00
Advertisements	6.00
Personal contributions	167.72
TOTAL	$466.07

*This does not include amounts forwarded to Editor Sanders by Secretary Cole covering pledges redeemed by different individuals to wipe out the 1927-28 News Letter debt. All such amounts were applied to said debt and properly credited on the Secretary's records and that debt is paid.

Since the above financial statement was made before the section at its Lafayette meeting Secretary W. D. Cairns has forwarded to Editor Sanders a check for fifty dollars ($50.00) as a generous tender by the Mathematical Association of America toward the "Expenses of the La.-Miss. Section for 1928-29." An accompanying note from Professor Slaught of the Finance Committee suggested its application to News Letter expenses. Needless to say, this keen and sympathetic interest on the part of the parent organization in the various sections which are affiliated with it is the greatest possible stimulus to greater efforts on the part of our mathematical workers.

PROBLEMS

Proposed by W. PAUL WEBBER

1. A ladder 20 ft. long stands vertically against a wall. A cat starts up the ladder and at the same time the foot of the ladder is pulled away from the wall at the same rate that the cat climbs. What will be the greatest elevation from the ground the cat can attain while the ladder is brought to a horizontal position on the ground?
2. A circular garden 100 ft. in diameter is fenced. A horse is tied to the fence on the outside of the garden by a rope 100 ft. long. Over how much ground can the horse graze?

This issue of the News Letter closes Vol. 3. The first issue of Vol. 4 will be the September number. We print eight issuses per year. Sample copies of the Letter for use in the canvass for subscribers may be had by writing to the editor-in-chief.